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# Finite noncommutative Chern-Simons with a Wilson line and the quantum hall effect

**Bogdan Morariu**

*Department of Physics, Rockefeller University*
*New York, NY 10021*
*E-mail:* morariu@summit.rockefeller.edu

**Alexios P. Polychronakos**[*]

*Department of Physics, Rockefeller University*
*New York, NY 10021*
*E-mail:* poly@theorfys.uu.se

Abstract: We present a finite dimensional matrix model associated to the noncommutative Chern-Simons theory, obtained by inserting a *Wilson* line. For a specific choice of the representation of the Wilson line the model is equivalent to minimal modification of the matrix model which is compatible with finite dimensional matrices, and was introduced previously to study droplets of quantum Hall fluid. For other representations we obtain generalizations corresponding to regularized $\mathrm{U}(n)$ Chern-Simons theories, representing multilayered quantum Hall fluids.

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[*]On leave from Theoretical Physics Dept., Uppsala University, Sweden and Physics Dept., University of Ioannina, Greece.

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## 1. Introduction

Recently, Susskind [1] proposed that the coordinates of electrons moving in a strong magnetic field $B$ be described by matrices, similar to the description of $D0$ brane coordinates. The corresponding lowest Landau level action, consisting of only the magnetic term [2], becomes a noncommutative version of the Chern-Simons action, which has found numerous applications in physics [3]. In the present context, it describes Laughlin fractional quantum Hall states [4], the Chern-Simons level becoming the inverse filling fraction. For a recent introduction to the quantum Hall effect see [5].

The above Chern-Simons theory can describe only an infinite number of electrons. In a previous paper, one of us (A.P.) proposed a regularized version of the noncommutative theory on the plane in the form of a Chern-Simons matrix model with boundary terms [6]. This model describes a system of finitely many electrons (a quantum Hall 'droplet') and reproduces all the relevant physics of the finite Laughlin states, such as boundary excitations [7, 8], quantization of the filling fraction (see also [9, 10]) and quantization of the charge of quasiparticles (fractional holes). An extension of this model for electrons on a cylinder, involving unitary matrices, is also introduced in [11]. An explicit, but non-unitary, mapping between the states of the matrix model and Laughlin states was presented in [12], while possible wavefunction mappings were explored in [13].

An essential ingredient of the above models is the so-called 'boundary' term. Its role is to absorb the anomaly of the theory and allow a finite matrix representation of the Gauss' law. In practice, it 'feeds' the representation of the gauge group U($N$) carried by the matrix coordinates of the model. This representation is fixed by the filling fraction and determines the physics of the electrons; it is the $(n-1)N$-fold symmetric representation of U($N$) [14], with the integer $n = \nu^{-1}$ representing the inverse filling fraction.

A natural question is whether other boundary terms could have been chosen and, correspondingly, different representations for U($N$). To approach this, we observe that the boundary term essentially corresponds to a Wilson line operator (see also [11]) and that a natural expression for such Wilson lines is through a first order gauge invariant action on the group manifold [15]. We therefore propose to use such a Wilson line action as the boundary term of the matrix model and study its classical and quantum structure.

The layout of this paper is as follows. In section 2 we review the noncommutative Chern-Simons and its finite dimensional cousin. Section 3 describes the proposed generalization of the action of [6]. Quantization of this action is equivalent to the insertion of a Wilson line. In section 4 we show how our proposal relates to that of [6]. We discuss both the classical and quantum equivalence. In the last section we discuss the quantum Hall interpretation and possible applications of the model. Finally, in order to fix our conventions and to make the paper relatively self-contained, we have included a brief description of the quantization of the symplectic form on coadjoint orbits in an appendix.

## 2. Review of noncommutative Chern-Simons in the quantum Hall effect

In Susskind's proposal, the dynamics of a system of $N$ electrons in the large $N$ limit is described by the action

$$\mathcal{S}_{NCCS} = \int dt \frac{B}{2} \operatorname{Tr} \{\epsilon_{ij}(\dot{X}^i + i[A_0, X^i])X^j + 2\theta A_0\} , \tag{2.1}$$

where $\theta$ is related to the average planar electron density $\rho_0 = 1/2\pi\theta$. The subscript $NCCS$ stands for noncommutative Chern-Simons and will be explained shortly.

Gauss' law obtained by varying $A_0$ in the action (2.1) takes the Heisenberg algebra form

$$[X^1, X^2] = i\theta . \tag{2.2}$$

Let $X^i = y^i$ be the solution of the Gauss' law equation (2.2) corresponding to $y^1 = x$ and $y^2 = \theta p$ where $x$ and $p$ are the matrices representing the position and momentum operators in the harmonic oscillator basis. If we insert $X^i = y^i + \epsilon^{ij}\theta A_j$ into the

the action (2.1), where $A_i$ parameterizes perturbations around the $y^i$ solution, one obtains the U(1) noncommutative Chern-Simons action in the operator language [16]

$$\mathcal{S}_{NCCS} = \frac{k}{4\pi} \int dt\, \epsilon^{\mu\nu\xi}\, 2\theta\, \mathrm{Tr}\left( A_\mu [\partial_\nu, A_\xi] + \frac{2}{3} A_\mu A_\nu A_\xi \right).$$

Here $\partial_i \equiv i\theta^{-1}\epsilon_{ij} y^i$ and is regarded as a (matrix) operator. The level $k$ of the Chern-Simons action equals (classically) the inverse filling fraction $\nu^{-1}$ where the filling fraction $\nu$ is defined as

$$\nu = \frac{2\pi\rho_0}{B}.$$

Taking the trace of (2.2) we see that Gauss' law is only compatible with infinite dimensional matrices. In order to describe a finite number of electrons, one of the authors [6], proposed a minimal addition to the action (2.1) to make it compatible with finite dimensional matrices. The proposed action is

$$\mathcal{S} = \mathcal{S}_{NCCS} + \mathcal{S}_\Psi + \mathcal{S}_\omega\,, \tag{2.3}$$

where

$$\mathcal{S}_\Psi = \int dt \Psi^\dagger (i\dot{\Psi} - A_0 \Psi)\,, \tag{2.4}$$

$$\mathcal{S}_\omega = -\int dt B\, \mathrm{Tr}\, \{\omega (X^i)^2\}\,. \tag{2.5}$$

The action (2.4) involves a complex bosonic N-vector $\Psi$ such that we have a U($N$) gauge invariance

$$\Psi \to U\Psi\,, \qquad X^i \to U X^i U^{-1}\,, \qquad \partial_t + iA^0 \to U(\partial_t + iA_0)U^{-1}\,.$$

The term (2.5) is just a spatial regulator, a harmonic potential whose role is to keep the electrons close to the origin. One can take $\omega$ arbitrarily small.

The $A_0$ equation of motion is now modified and reads

$$iB\,[X^1, X^2] - \Psi\Psi^\dagger + B\theta = 0\,. \tag{2.6}$$

The trace of (2.6) then simply implies

$$\Psi^\dagger\Psi = Nk\,, \tag{2.7}$$

which is compatible with finite dimensional matrices. It was shown in [6], using the equivalence to the Calogero model, that upon quantization this system describes Laughlin states for the Quantum Hall effect of $N$ electrons at fractional filling fraction $\nu = 1/(k+1)$. For a recent discussion of the equivalence of (2.3) and the Calogero model see [17] and references there. In the $A_0 = 0$ gauge, one can first quantize the system ignoring the constraint (2.6) and then impose it on physical states. In the absence of the constraint we have a system of free harmonic oscillators, some coming from the matrix elements of $X^i$ and some from the components of $\Psi$. Gauss' law (2.6) implies that physical states are invariant under U($N$) gauge transformations[1]. A group theoretical analysis of this physical state condition was used in [12] to find a more direct mapping to Laughlin states.

## 3. The Wilson line action

In this section we propose a generalization of the action (2.3). We propose to replace $\mathcal{S}_\Psi$ with with

$$\mathcal{S}_g = \int dt\, \mathrm{Tr}\left[i\lambda g^{-1}(\partial_t + iA_0)g\right], \tag{3.1}$$

where $g$ is valued in the U($N$) group and $\lambda$ is an arbitrary hermitian matrix [15]. Without loss of generality $\lambda$ can be taken to be diagonal. Let $H$ denote the stability group of $\lambda$ under the adjoint action of U($N$). For generic $\lambda$ the subgroup $H$ is just the diagonal Cartan subgroup.

This system is gauge invariant under the transformations $g' = gh$, where $h$ is time dependent and $H$ valued. In fact, the action (3.1) is invariant under infinitesimal transformations. For large gauge transformations the action changes by an integer multiple of $2\pi$ only if $\lambda$ is quantized. The gauge invariant degrees of freedom are points on the coadjoint orbit of $G$ passing through $\lambda$. These are symplectic leaves, and it is a well known fact [18] that their quantization gives the unitary representations of the Lie group $G$. The quantization of this system for $g$ valued in a simple Lie group $G$ is reviewed in the appendix. The method there, is similar in spirit to the one used in [19]. See also [20] for a path integral quantization, using Darboux coordinates. Here we only explain the necessary modifications for

$$\mathrm{U}(N) \cong (\mathrm{SU}(N)\times \mathrm{U}(1))\,/\,\mathbb{Z}_N\,.$$

First let us obtain the quantization of $\lambda$. Let the Cartan generators $H_i$ of SU($N$) be

$$H_i = \mathrm{diag}(0,\ldots,0,1,-1,\ldots,0)\,, \qquad i = 1,\ldots,N-1\,, \tag{3.2}$$

where the only nonvanishing entries are on the $i$ and $i+1$ positions. To this we append $H_N = \mathbb{I}$ so that we have a complete set in the Cartan subalgebra of U($N$). Similarly, let the fundamental weights of su($N$) be given by[2]

$$\mu^i = \frac{1}{N}\,\mathrm{diag}(N-i,\ldots,N-i,-i,\ldots,-i)\,, \qquad i = 1,\ldots,N-1\,,$$

[1] We have included the background charge $B\theta$.

[2] Since the trace is nondegenerate we will identify the Lie algebra with its dual. The same can be done for the Cartan subalgebra and its dual.

where the first $i$ and the last $N-i$ diagonal entries of $\mu^i$ are equal. To this set we add $\mu^N = 1/N\,\mathbb{I}$ such that we now have $\mathrm{Tr}\,(H_i\,\mu^j) = \delta_i^j,\ i = 1,\ldots,N$. Using this we can write $\lambda = -n_i\mu^i$ where $n_i = -\mathrm{Tr}(\lambda H_i)$. Consider now a large gauge transformation of the form

$$h(t) = e^{2\pi i H_i \xi^i(t)}\,,$$

where $\xi^i$ are real functions of time such that $h(t) = \mathbb{I}$ for large positive or negative times. This periodicity implies that

$$H_i \Delta\xi^i = \mathrm{diag}(k_1,\ldots,k_N)\,, \tag{3.3}$$

where the diagonal elements $k_i$ must be integers. Under the gauge transformation $h(t)$ the lagrangian changes by a total derivatives which can be integrated to give the following change of the action

$$\Delta\mathcal{S}_g = 2\pi\mathrm{Tr}\left[-\lambda(H_i\Delta\xi^i)\right] = 2\pi n_i \Delta\xi^i\,.$$

The requirement that the action only changes by an integer multiple of $2\pi$ together with (3.3) implies that all the diagonal elements of $\lambda$ must be integers. Then $n_i = -\mathrm{Tr}(\lambda H_i)$ are also integers. For arbitrary $h \in H$, one can show that $\lambda_i$, the diagonal matrix elements of $\lambda$ satisfy

$$\lambda_i = -\frac{1}{N}\left[\sum_{i=1}^{N-1}(N-i)n_i + n_N\right] + 0 \quad (\mathrm{mod}\ 1)\,.$$

Thus, to be free of global gauge anomalies $n_i$ must all be integers and satisfy

$$\sum_{i=1}^{N-1}(N-i)n_i + n_N = 0 \quad (\mathrm{mod}\ N)\,. \tag{3.4}$$

The quantization can be now be performed similarly to the treatment in the appendix of a simple Lie group. Here we will be brief and only outline the steps (see the appendix for details). The same dynamics can be obtained from a nondegenerate first order action

$$\mathcal{S} = \int dt\,\mathrm{Tr}\left[ipg^{-1}\dot{g} - gpg^{-1}A_0\right], \tag{3.5}$$

together with the constraint

$$p - \lambda = 0\,. \tag{3.6}$$

The symplectic structure obtained from the first term in (3.5) is the standard one on the cotangent bundle of $\mathrm{U}(N)$. The Hilbert space of the unconstrained system is given by square integrable functions on $\mathrm{U}(N)$ whose harmonic expansion is given by Peter-Weyl theorem. Just as in the appendix, one then imposes only the constraints (3.6) associated with the Cartan and positive roots generators. The physical Hilbert space



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is then finite dimensional and provides an irreducible representation of U($N$). It is the representation whose *lowest* weight is given by $\lambda$. Note that the first term on the left hand side of (3.4) is just the number of boxes in the Young tableau of the SU($N$) representation, while the second is the U(1) charge.

Finally we can now explain the title of this section. The path integral over $g$ in the action (3.1) gives a Wilson line in the irreducible representation discussed above

$$\int dg\, e^{i\mathcal{S}_g} = \mathrm{P} e^{i\int dt\, A_0^a(t) t_a}\,, \tag{3.7}$$

where P denotes path ordering. Both the right and the left hand side of (3.7) are understood as operators acting in the physical Hilbert space. This can be seen as follows. Using the notation in the appendix we can write

$$\mathrm{Tr}(\lambda g A_0 g^{-1}) = \tilde{p}_a A_0^a\,.$$

Upon quantization as can be seen from (A.4), $-\tilde{p}_a$ becomes the operator acting on the Hilbert space as $t_a$. Since the relation between operators and path integral also involves time ordering, which is path ordering in this case, we obtain the desired result.

Gauss' law for the full action $\mathcal{S} = \mathcal{S}_{NCCS} + \mathcal{S}_g + \mathcal{S}_\omega$ now reads

$$iB\,[X^1, X^2] - g\lambda g^{-1} + B\theta = 0\,, \tag{3.8}$$

and the trace of (3.8) implies a constraint on the trace of $\lambda$

$$\mathrm{Tr}(\lambda) = Nk\,. \tag{3.9}$$

## 4. Relation to boundary fields

In this section we will demonstrate the correspondence of this model with the previous model with boundary $\Psi$ terms. Specifically, we will show that the Wilson line action involving $g$ is a particular sector of a version of the previous model involving $N$ different boundary fields $\Psi$.

First let us discuss the classical equivalence. The U($N$) matrix $g$ can be written in terms of its columns $\psi_j$, $j = 1, 2, \ldots N$. Since $g^{-1} = g^\dagger$, the action $\mathcal{S}_g$ becomes

$$\mathcal{S}_g = \int dt \sum_j \left[ i\lambda_j\, \psi_j^\dagger (\partial_t + iA_0)\psi_j \right]. \tag{4.1}$$

We have the sum of $N$ decoupled actions. The vectors $\psi_j$ are still coupled through the relations

$$\psi_j^\dagger \psi_k = \delta_{jk}\,, \tag{4.2}$$

implied by the unitarity of $g$. Upon imposing Gauss' law, however, and putting $A_0 = 0$ the equations of motion for $g$ are

$$[\lambda, g^{-1}\dot{g}] = 0\,, \tag{4.3}$$

and in terms of $\psi_j$ they become

$$(\lambda_j - \lambda_k)\,\psi_j^\dagger\dot{\psi}_k = 0 \qquad (\text{no sum in } j,k)\,. \tag{4.4}$$

For generic values of $\lambda_j$ the above equations imply that $\dot{\psi}_j = 0$ is orthogonal to all other $\psi_k$, and therefore $\dot{\psi}_j \sim \psi_j$. Since $\psi_j$ is normalized, this means that only its phase can vary. The Wilson line action (3.1), on the other hand, has an additional $\mathrm{U}(1)^N$ gauge invariance, corresponding to right-multiplication of $g$ by an arbitrary diagonal unitary matrix. This transformation is exactly the redefinition of the phases of $\psi_j$. So the above motion of $\psi_j$ corresponds to a $\mathrm{U}(1)^N$ gauge transformation and, as a gauge choice, we can take $\dot{\psi}_j = 0$. These are the same equations of motion that we would have derived from the action (4.1), ignoring the constraints (4.2). The constraints, then, can be imposed as initial data for the $\psi_j$.

We can go one step further and redefine the $\psi_j$ to absorb $\lambda_j$. We distinguish between the cases $\lambda_j < 0$ and $\lambda_j > 0$. We define

$$\Psi_j = \sqrt{|\lambda_j|}\,\psi_j\,. \tag{4.5}$$

Assuming that there are $n$ positive and $N-n$ negative $\lambda$'s the action becomes

$$\mathcal{S}_g = \int dt \sum_{j=1}^{n}\left[\Psi_j^\dagger(i\partial_t - A_0)\Psi_j\right] + \sum_{j=n+1}^{N}\left[\Psi_j^\dagger(-i\partial_t + A_0)\Psi_j\right]. \tag{4.6}$$

This is identical to the original action $\mathcal{S}_\Psi$, where, now, we have introduced a multiplet of boundary fields $\Psi_j$, with $n$ of them transforming under the fundamental of the gauge group and $N-n$ transforming under the anti-fundamental representation. We must further choose as initial conditions

$$\Psi_j^\dagger\Psi_k = |\lambda_j|\,\delta_{jk}\,. \tag{4.7}$$

So we have traded $\lambda$ for the initial conditions of the lengths of $\Psi_j$, and we have the additional condition of orthogonality between the different $\Psi_j$. This last requirement is not a restriction. The generator of gauge transformations for the $\Psi_j$, which enters Gauss' law, is

$$G_\Psi = \sum_{j=1}^{n}\Psi_j\Psi_j^\dagger - \sum_{j=n+1}^{N}\Psi_j\Psi_j^\dagger\,. \tag{4.8}$$

This is a hermitian matrix which obviously projects in the space spanned by $\Psi_j$, so it can be diagonalized by a unitary transformation in this space. This amounts to a linear redefinition of the $\Psi_j$ such that they be orthogonal to each other. These redefined $\Psi_j$ satisfy (4.7).

If any of the $\lambda_j$ are equal to each other then the equations of motion (4.4) do not imply that $\dot{\psi}_j$ is proportional to $\psi_j$, for the corresponding values of $j$, but rather to an arbitrary linear combination of these $\psi_j$'s. In this case, however, the Wilson line action has an enhanced gauge invariance under right-multiplications of $g$ by unitary matrices in the corresponding subspace. The equations of motion then allow an arbitrary motion in this subspace, which becomes a gauge transformation. It is consistent, therefore, to also impose $\dot{\psi}_j = 0$ in this case, which amounts to a gauge choice. The gauge generator $G_\Psi$ in (4.8) in this case will have a degenerate subspace for the corresponding $j$, and the gauge arbitrariness corresponds to the freedom of rotating the $\Psi_j$ in this subspace.

We conclude that in all cases the Wilson line action $\mathcal{S}_g$ is essentially equivalent to the action $\mathcal{S}_\Psi$ for $N$ independent boundary fields $\Psi_j$ transforming in the (anti)fundamental of $\mathrm{U}(N)$, after fixing the initial conditions and choosing the basis where the matrix generator $G_\Psi$ in the Gauss' law is already diagonalized. The corresponding theories with many boundary fields are similar to the models in [21, 22] which, upon proper reduction, lead to spin-Calogero models.

Quantum mechanically a similar picture emerges. In the theory with $n$ boundary fields in the fundamental and $N-n$ boundary fields in the antifundamental the components of the fields $(\Psi_j)_k$ for $j \leq n$, and $(\Psi_j)_k^\dagger$ for $j > n$ (where $\dagger$ now stands only for operator, not matrix, hermitian conjugation) become oscillator annihilation operators (note the opposite sign of the canonical term in (4.6) for $j > n$). The boundary gauge generators are the sum of $N$ independent oscillator realizations of the generators of $\mathrm{U}(N)$. Renaming $(\Phi_j)_k = (\Psi_j)_k^\dagger$ we have the ordering

$$G_\Psi^a = \sum_{k=1}^{n} (\Psi_k)_i^\dagger T_{ij}^a (\Psi_k)_j - \sum_{k=n+1}^{N} (\Phi_k)_i^\dagger T_{ji}^a (\Phi_k)_j \,. \tag{4.9}$$

The Fock spaces of the $\Psi, \Phi$ embeds the tensor product of the representations of each oscillator realization; for $j \leq n$ the oscillator $\Psi_j$ reproduces all totally symmetric irreps of $\mathrm{SU}(N)$ (all irreps with a single row in their Young tableau), with $\mathrm{U}(1)$ charge equal to the total number operator. For $j > n$ the oscillator $\Phi_j$ reproduces the conjugates of the above representations (irreps with $N-1$ rows of equal length). The tensor product of $N$ such irreps of all possible lengths contains all representations of $\mathrm{SU}(N)$. Therefore, the model with $N$ boundary fields allows for the most general representation of the Gauss' law generator. Picking a particular irrep is the quantum mechanical analog of fixing initial conditions and diagonalizing the classical gauge generator matrix. The action $\mathcal{S}_g$, on the other hand, also reproduces an arbitrary representation of the Gauss' law generator, upon picking the appropriate $\lambda$.

We conclude that the two theories are essentially equivalent, with the difference that $\mathcal{S}_g$ picks an irreducible component for the Gauss' law generator, that is, a particular sector of the $\mathcal{S}_\Psi$ theory. The original single-$\Psi$ theory only has one sector



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and it is completely reproduced by the Wilson line model with all but a single of the $\lambda_j$'s vanishing:

$$\lambda = -Nk\mu^{N-1} + Nk\mu^N = \mathrm{diag}(0, \ldots, 0, Nk)\,. \tag{4.10}$$

Notice that the U(1) charge of the representation, determined by tr$\lambda$, corresponds to the total number operator of the oscillators. By adding such a U(1) part we could render all $\lambda_j$ positive and dispense with the anti-fundamental fields in the $\Psi$ representation. We stress, however, that the total U(1) charge essentially determines the noncommutativity parameter $\theta$ (the filling fractions) through the Gauss' law. Including, therefore, some antifundamental fields is crucial is we wish to reproduce all irreps of SU($N$) for a fixed value of $\theta$.

## 5. Discussion

The proposed generalization allows for the maximal flexibility in the choice of the representation for the Gauss' law for the matrix coordinates $X_i$. The Wilson line ($g$) representation is particularly convenient in that it picks a single irreducible representation. The boundary field ($\Psi_j$) representation, on the other hand, is most suited to discuss the physics of the model.

One obvious application of the many-$\Psi$ model would be to describe many layers of quantum Hall fluids and/or fluids with spin. Intuitively, the presence of many boundary fields creates many boundaries for the quantum Hall droplet which, then, decomposes into many layers. The total number of electrons in all layers is always $N$.

As an example, consider the case of two fields $\Psi_{1,2}$, both in the fundamental, satisfying

$$\Psi_i^\dagger \Psi_j = kN_i\delta_{ij}\,, \qquad N_1 + N_2 = N\,, \tag{5.1}$$

where $N_{1,2}$ are integers. Then the Gauss' law implies

$$iB[X^1, X^2] + k\,\mathrm{diag}\,(1, \ldots 1 - N_1, 1, \ldots 1 - N_2) = 0\,. \tag{5.2}$$

We have chosen a basis in which the Gauss' law is diagonal, with the entry $1 - N_1$ appearing in the position $N_1$ in the diagonal. In this way the trace of the first $N_1$ elements of the above matrix vanishes. So (5.2) admits block-diagonal solutions for $X_1$ and $X_2$, each representing a quantum Hall droplet with $N_1$ or $N_2$ electrons. The two droplets can obviously overlap. So this model can describe different quantum Hall layers as well as their interactions (non-block diagonal solutions). The generalization to $n$ layers is straightforward. The layers can, equivalently, be viewed as spin components for the electrons. The relation of the $n$-component model to the SU($n$)-spin Calogero model [21] further enhances the likelihood of such a correspondence, although the details are yet to be worked out.

Another obvious application of the proposed model is as a regularization of U($n$) noncommutative Chern-Simons theory. For the infinite plane case the Gauss' law (2.2) can admit reducible representations corresponding to the direct sum of $n$ Heisenberg representations. Perturbations around such a solution would give rise to U($n$) NCCS action. The single-boundary matrix model, on the other hand, has a ground state corresponding to a single layer which reproduces U(1) NCCS theory. By choosing, however, $n$ boundary fields in the fundamental and taking them to be orthogonal and of norm squared $kN/n$ ($N$ should be a multiple of $n$) we have a situation analogous to the one above, which admits as ground state a block-diagonal configuration which is the direct sum of $n$ ground states of size $N/n$ each. Clearly there is an extra U($n$) symmetry mixing the $n$ components. Perturbations around this configuration would give a regularized version of U($n$) theory.

In conclusion, the extended model proposed here has many potential applications both in noncommutative gauge theory and in the quantum Hall context, which are the topic of further investigation.

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## A. Coadjoint orbits quantization

In this appendix we describe the quantization of the system defined by the action

$$\mathcal{S} = \int dt \,\mathrm{Tr}\left[i\lambda g^{-1}(\partial_t + iA_0)g\right], \tag{A.1}$$

where $g$ is valued in the group $G$ which we assume to be a simple Lie group. See [15] and references there, and for a recent treatment see [19]. The normalization of the trace is chosen such that the length square of a long root equals two. For $G = \mathrm{SU}(N)$ this reduces to the matrix trace in the defining representation. In (A.1) $\lambda$ is a constant weight.

The action (A.1) is equivalent to the action

$$\mathcal{S} = \int dt \,\mathrm{Tr}\left[ipg^{-1}\dot{g} - gpg^{-1}A_0\right], \tag{A.2}$$

together with the constraint

$$p - \lambda = 0\,. \tag{A.3}$$

The first term in the action (A.2) gives the standard symplectic form on the cotangent bundle of the group $G$

$$\omega = d\,\mathrm{Tr}(ipg^{-1}dg)\,,$$

from which we can derive the Poisson bracket

$$i\,\{f,h\} = \mathrm{Tr}\left(\frac{\partial f}{\partial g^T} g \frac{\partial h}{\partial p^T} - \frac{\partial h}{\partial g^T} g \frac{\partial f}{\partial p^T} + \frac{\partial f}{\partial p^T} p \frac{\partial h}{\partial p^T} - \frac{\partial h}{\partial p^T} p \frac{\partial f}{\partial p^T}\right).$$

Let $p_a = -\mathrm{tr}(t_a p)$ and $g_{MN}$ be coordinates on the cotangent bundle. They have the following Poisson brackets

$$i\{p_a, p_b\} = i f_{ab}{}^c\, p_c\,, \qquad i\{p_a, g_{MN}\} = (g t_a)_{MN}\,,$$

which upon quantization become

$$[p_a, p_b] = i f_{ab}{}^c\, p_c\,, \qquad [p_a, g_{MN}] = (g t_a)_{MN}\,.$$

Let $\tilde{p} = gpg^{-1}$ designate the matrix in front of $A_0$ in the action (A.2). Then the operators $\tilde{p}_a = \mathrm{tr}(t_a \tilde{p})$ commute with $p_a$ and satisfy

$$[\tilde{p}_a, \tilde{p}_b] = i f_{ab}{}^c\, \tilde{p}_c\,, \qquad [\tilde{p}_a, g_{MN}] = (-t_a g)_{MN}\,. \tag{A.4}$$

We can represent the Hilbert space $\mathcal{H}$ of the unconstrained system using square integrable functions on the group manifold. By Peter-Weyl theorem any function on the group has the decomposition

$$\psi(g) = \sum_R \sum_{\alpha\beta} c_R^{\alpha\beta}\, R_{\alpha\beta}(g)\,, \tag{A.5}$$

where $R_{\alpha\beta}(g)$ are the matrix elements of the $R$ representation of $G$. They satisfy the following orthogonality conditions

$$\int dg\, R_{\alpha\beta}(g)\, \bar{R}'_{\rho\sigma}(g) = \frac{1}{d_R} \delta_{RR'}\, \delta_{\alpha\rho}\, \delta_{\beta\sigma}\,,$$

where $dg$ is the the Haar measure and $d_R$ is the dimension of the $R$ representation.

The operators $p_a$ and $\tilde{p}_a$ act on this Hilbert space generating an action of $G_l \times G_r$, where we used a subscript to distinguish the two $G$ factors. Under the action of $(g_l\,, g_r) \in G_l \times G_r$ we have the following transformation

$$\psi(g) \to \psi'(g) = \psi(g_l^{-1}\, g\, g_r)\,. \tag{A.6}$$

In particular $R_{\alpha\beta}(g)$ transforms as

$$R_{\alpha\beta}(g) \to R_{\rho\sigma}(g) \bar{R}_{\rho\alpha}(g_l) R_{\sigma\beta}(g_r)\,,$$

that is, the index $\beta$ transforms in the representation $R$ while the index $\alpha$ transforms in the representation $\bar{R}$. We can reinterpret (A.5) as the decomposition of the Hilbert space $\mathcal{H}$ under the action (A.6) into the following sum of irreducible representations

$$\mathcal{H} \cong \bigoplus_R V_{\bar{R}} \otimes V_R \,. \tag{A.7}$$

Here $V_R$ is the vector space in which the representation $R$ acts, $\bar{R}$ is the complex conjugate representation and the sum is over all the inequivalent unitary irreducible representations of $G$. If the states $|\alpha, R\rangle$, $\alpha = 1, \ldots, d_R$ form a basis of $V_R$ and $|\overline{\alpha, R}\rangle$ a basis of $V_{\bar{R}}$, the isomorphism is given by

$$R_{\alpha\beta}(g) \to |\overline{\alpha, R}\rangle \otimes |\beta, R\rangle \,.$$

Let us now consider the constraint (A.3). Using the raising and lowering generators $e_\alpha$ and $e_{-\alpha}$, where $\alpha$ are positive roots (not to be confused with the index labeling the components of representations), and $H_i$ forming a basis in the Cartan subalgebra[3] we define

$$\begin{aligned} p_i &= -\operatorname{tr}(H_i\, p)\,, \qquad i = 1, \ldots, \operatorname{rank}(G)\,, \\ p_\alpha &= -\operatorname{tr}(e_\alpha\, p)\,, \qquad p_{-\alpha} = -\operatorname{tr}(e_{-\alpha}\, p)\,, \qquad \alpha > 0\,. \end{aligned}$$

Using $\operatorname{tr}(H_i\, \mu^j) = \alpha_i^\vee(\mu^j) = \delta_i^j$ we can rewrite the constraint (A.3) as

$$p_i - n_i = p_\alpha = p_{-\alpha} = 0\,, \qquad \alpha > 0\,, \tag{A.8}$$

where $n_i = -\operatorname{tr}(H_i \lambda)$. At the classical level, some of these constraints are first class and some are second class. To see this, note that for some positive roots $\alpha$ the Poisson bracket $\{p_\alpha, p_{-\alpha}\}$ gives linear combinations of the Cartan $p_i$ which can be nonvanishing due to the constraints $p_i = n_i$. At the quantum level we can not impose all the constrains simultaneously. We can however use a Gupta-Bleuer type quantization and require that physical states $\psi$ satisfy only

$$\begin{aligned} p_i\, \psi(g) &= n_i\, \psi(g)\,, \\ p_\alpha\, \psi(g) &= 0\,, \qquad \alpha > 0\,. \end{aligned} \tag{A.9}$$

These are the defining relations for a highest weight state which only exists in the decomposition (A.7) if $n_i$ are positive integers. This is how the quantization of the weight $\lambda$ is obtained in the hamiltonian approach. Physical states must therefore have the form

$$|\overline{\beta, R}\rangle \otimes |\beta_0, R\rangle \quad \beta = 1, \ldots, d_R\,,$$

[3]Using the trace, $H_i$ can be identified with the simple coroots $\alpha_i^\vee$.

where $|\beta_0, R\rangle$ is the highest weight state of weight $-\lambda$. In the wave function on the group description we are restricted to functions of the form

$$\psi(g) = \sum_{\beta} c^{\beta} R_{\beta\beta_0}(g)\,.$$

The physical Hilbert space is the irreducible representation of $G_l$ whose complex conjugate representation highest weight is $-\lambda$. Equivalently, the physical Hilbert space is the irreducible representation whose *lowest* weight is $\lambda$.

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